Dynamics of Gyroelastic Spacecraft

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This paper introduces the concept of a gyroelastic spacecraft—a vehicle comprising not only a continuous distribution of mass and elasticity but a continuous distribution of gyricity (stored angular momentum) as well. It is assumed that the spacecraft has a number of gyroelastic appendages and that the (constrained) mode shapes are known for each of these appendages. General vehicle deformations are expanded in terms of these mode shapes. The eigenproblem for the (unconstrained) vehicle reveals a significant departure from the modal behavior of nongyric elastic vehicles. In particular, a gyroelastic vehicle can exhibit a "pseudorigid" mode in which the vehicle rotates uniformly in a deformed state. Although the frequency of this mode is zero, the associated strain energy is nonzero, and it is therefore not a "rigid" mode.

Introduction

THE concept of a gyroelastic continuum, that is, a body considered to be continuous not only in mass and stiffness but in gyricity as well, was introduced in a recent paper.[1] The notion of using a continuous gyricity distribution to represent, as a limiting case, a large number of rotors spread over an elastic structure—as momentum wheels and/or control moment gyros might be distributed over a large flexible spacecraft for control purposes—was found to be a particularly useful analytical tool. (Discrete rotors are included as a special case by using Dirac delta functions at discrete points in the continuous distribution.) The earlier paper examined the dynamics of gyroelastic bodies constrained (cantilevered) to a rigid base. Such bodies do not exhibit "rigid" modes. A set of modal coefficients associated with (constrained) gyroelastic continua was presented in a subsequent study.[2] These coefficients express the momentum and angular momentum corresponding to each mode, and it was shown that they satisfy a number of interesting and useful modal identities.

The analysis of constrained gyroelastic dynamics has a practical dimension in itself; it is applicable to systems that possess a gyric character and are positive-definite in strain energy. But, more important, it lays the foundation for the study of *unconstrained* gyroelastic dynamics, that is, the motion of gyroelastic vehicles. The analysis of unconstrained dynamics may be approached from two different directions: 1) by expressing the unconstrained motion in terms of constrained dynamical parameters, i.e., the modal coefficients; or 2) by formulating the unconstrained problem *ab initio* and proceeding in the same manner as in the constrained dynamical analysis. The present paper is devoted to the former approach, while a companion paper[3] deals with the latter approach. If the appendages are specified separately in terms of their gyroelastic modes, then the present formulation is applicable. If the spacecraft is specified as a whole, or if the spacecraft is one large flexible structure, the approach in Ref. 3 is preferable.

The analysis to follow encompasses all flexible spacecraft that are (nonspinning) three-axis-controlled. The flexible portions of the spacecraft are taken to be linear-elastic. In addition, we shall assume that all the periods of the spacecraft's structural vibrations are much shorter than the orbital period. This affords us the convenience of taking the local orbiting reference frame as approximately inertial for the equations of motion.[4] Moreover, the orbital and attitude motions of the spacecraft will be assumed uncoupled, to first order. Both of these assumptions, however, can be relaxed, provided the necessary additional terms are included in the motion equations.

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Gyroelastic Vehicle

Consider the vehicle V in Fig. 1 consisting of a number of flexible appendages, collectively denoted by E, attached to a rigid body R. Let O represent an origin fixed in R. We assume[1] a gyricity distribution $\boldsymbol{h}_s(\boldsymbol{r})$ over the vehicle (including R) that is constant with respect to the local reference frame at $\boldsymbol{r}$. The static elastic deformation $\boldsymbol{u}(\boldsymbol{r})$ of the vehicle is related to the static force distribution $\boldsymbol{f}(\boldsymbol{r})$ acting on it via the linear *stiffness operator* $\mathscr{K}$:

$$\mathscr{K}\boldsymbol{u}=\boldsymbol{f} \tag{1}$$

It should be noted that this relationship can also include the effect of a torque distribution $\boldsymbol{g}(\boldsymbol{r})$ on the vehicle since an equivalent force distribution for $\boldsymbol{g}(\boldsymbol{r})$ is given[1] by

$$\boldsymbol{f}(\boldsymbol{r})=\tfrac{1}{2}\nabla^{\times}\boldsymbol{g}(\boldsymbol{r}) \tag{2}$$

Thus, the torques that arise from $\boldsymbol{h}_s(\boldsymbol{r})$, as well as from external sources, can be incorporated into Eq. (1). The cross notation defines a skew-symmetric matrix for an arbitrary column matrix, namely,

$$e^{\times}=\begin{pmatrix} \cdot & -e_3 & e_2 \\ e_3 & \cdot & -e_1 \\ -e_2 & e_1 & \cdot \end{pmatrix}, \qquad e=\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

and so $\nabla^{\times}$ is the matrix representation of the curl operator.

Since the vehicle is unconstrained, it possesses rigid degrees of freedom. Mathematically, this condition may be expressed as

$$\mathscr{K}\boldsymbol{u}_r=\boldsymbol{0} \tag{3}$$

where the rigid displacement $\boldsymbol{u}_r(\boldsymbol{r})$ is not identically zero. The corresponding modes are the three rigid-body translations and the three rigid-body rotations:

$$\boldsymbol{u}_r=\boldsymbol{1}_1,\boldsymbol{1}_2,\boldsymbol{1}_3,\boldsymbol{1}_1^{\times}\boldsymbol{r},\boldsymbol{1}_2^{\times}\boldsymbol{r},\boldsymbol{1}_3^{\times}\boldsymbol{r} \tag{4}$$

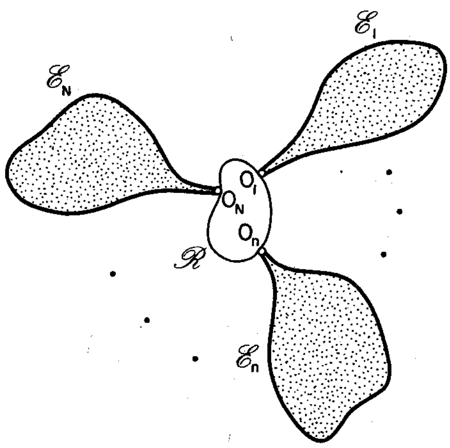

Fig. 1 Morphology of a gyroelastic spacecraft.

or, briefly,

$$U_r(\boldsymbol{r}) \triangleq [\mathbf{1} - \boldsymbol{r}^\times] \tag{5}$$

where $\mathbf{1}_1, \mathbf{1}_2, \mathbf{1}_3$ are the columns of $\mathbf{1}$, the identity matrix. For clarity, the following analysis does not consider *internal* rigid degrees of freedom, which a rotational joint, for example, would provide. The extension, however, is straightforward.[5]

Modal Coefficients

Let us briefly review some aspects of constrained gyroelastic dynamics. For an appendage E_n attached to a rigid base, the equation of motion in first-order operator form is

$$\mathscr{E}\dot{\chi} + \mathscr{S}\chi = \gamma_e \tag{6}$$

where

$$\mathscr{E} \triangleq \begin{pmatrix} \mathscr{M} & \cdot \\ \cdot & \mathscr{K} \end{pmatrix}, \qquad \mathscr{S} \triangleq \begin{pmatrix} \mathscr{G} & \mathscr{K} \\ -\mathscr{K} & \cdot \end{pmatrix}$$

$$\chi \triangleq \begin{pmatrix} \dot{\boldsymbol{u}} \\ \boldsymbol{u} \end{pmatrix}, \qquad \gamma_e \triangleq \begin{pmatrix} \boldsymbol{f}_{eT} \\ \cdot \end{pmatrix}$$

The dots imply a null entry and $\boldsymbol{f}_{eT}$ is the total external force field, which includes external torques:

$$\boldsymbol{f}_{eT} = \boldsymbol{f}_e + \tfrac{1}{2}\nabla^\times \boldsymbol{g}_e$$

The *mass* and *gyricity operators* are defined as

$$\mathscr{M} \triangleq \sigma(\boldsymbol{r})\,1, \qquad \mathscr{G} \triangleq -\tfrac{1}{4}\nabla^\times \boldsymbol{h}_s^\times \nabla^\times \tag{7}$$

where $\sigma(\boldsymbol{r})$ is the mass density of the vehicle and 1 the identity operator. For discrete systems, the operators $\mathscr{E}$ and $\mathscr{S}$ become matrices. The definitions of $\mathscr{E}$ and $\mathscr{S}$ here follow analogously to the definitions for discrete gyroelastic systems.[6]

The system eigenfunctions have the form

$$\phi_\alpha = \begin{pmatrix} -\Omega_\alpha \boldsymbol{v}_\alpha \\ \boldsymbol{u}_\alpha \end{pmatrix}, \qquad \psi_\alpha = \begin{pmatrix} \Omega_\alpha \boldsymbol{u}_\alpha \\ \boldsymbol{v}_\alpha \end{pmatrix} \tag{8}$$

where Ω_α is the vibration frequency, and the function pair $\{\boldsymbol{u}_\alpha, \boldsymbol{v}_\alpha\}$ describes the corresponding mode shape $(-\infty < \alpha < \infty,\ \alpha \neq 0)$. Without loss of generality, we can set $\Omega_{-\alpha} = -\Omega_\alpha$, $\phi_{-\alpha} = \psi_\alpha$, $\boldsymbol{u}_{-\alpha} = \boldsymbol{v}_\alpha$. Moreover, the eigenfunctions satisfy the following orthogonality conditions:

$$\int_{E_n} \phi_\alpha^T \mathscr{E} \phi_\beta \,\mathrm{d}V = 2\Omega_\alpha^2 \delta_{\alpha\beta}$$

$$\int_{E_n} \psi_\alpha \mathscr{S} \phi_\beta \,\mathrm{d}V = 2\Omega_\alpha^3 \delta_{\alpha\beta} \tag{9}$$

The general deformation of the appendage may be expanded in terms of ϕ_α:

$$\chi(\boldsymbol{r}, t) = \sum_{\alpha=-\infty}^{\infty} \phi_\alpha(\boldsymbol{r})\eta_\alpha(t) \tag{10}$$

The momentum and angular momentum (about O) of the appendage can then be written as

$$\boldsymbol{p}(t) = \sum_\alpha \boldsymbol{p}_\alpha \dot{\eta}_\alpha$$

$$\boldsymbol{h}(t) = \boldsymbol{h}_T + \sum_\alpha (\boldsymbol{h}_\alpha \dot{\eta}_\alpha + \boldsymbol{g}_\alpha \eta_\alpha) \tag{11}$$

where $\boldsymbol{h}_T$ is the total gyricity, and where

$$\boldsymbol{p}_\alpha \triangleq \int_{E_n} \boldsymbol{u}_\alpha(\boldsymbol{r})\,\mathrm{d}m \equiv \int_{E_n} \mathscr{M}\boldsymbol{u}_\alpha \,\mathrm{d}V$$

$$\boldsymbol{h}_\alpha \triangleq \int_{E_n} \boldsymbol{r}^\times \boldsymbol{u}_\alpha(\boldsymbol{r})\,\mathrm{d}m \equiv \int_{E_n} \boldsymbol{r}^\times \mathscr{M}\boldsymbol{u}_\alpha \,\mathrm{d}V$$

$$\boldsymbol{g}_\alpha \triangleq -\tfrac{1}{2}\int_{E_n} \boldsymbol{h}_s^\times(\boldsymbol{r})\nabla^\times \boldsymbol{u}_\alpha(\boldsymbol{r})\,\mathrm{d}V \equiv \int_{E_n} \boldsymbol{r}^\times \mathscr{G}\boldsymbol{u}_\alpha \,\mathrm{d}V$$

are, respectively, the *modal momentum*, *angular-momentum*, and *stored-momentum coefficients*.

Equations of Motion

The total displacement $\boldsymbol{w}(\boldsymbol{r}, t)$ of the vehicle V at $\boldsymbol{r}$ is the sum of the rigid displacement and the elastic displacement, i.e.,

$$\boldsymbol{w}(\boldsymbol{r}, t) = \boldsymbol{w}_0(t) - \boldsymbol{r}^\times \boldsymbol{\theta}(t) + \begin{cases} \boldsymbol{0} & \boldsymbol{r} \in R \\ \boldsymbol{u}(\boldsymbol{r}, t) & \boldsymbol{r} \in E \end{cases} \tag{12}$$

Parsing the expression, we have $\boldsymbol{w}_0$ as the translation of O, $\boldsymbol{\theta}$ the rotation of R (assumed small relative to an inertial frame), and $\boldsymbol{u}(\boldsymbol{r}, t)$ the elastic deformation of the appendages.

The motion of the appendages can be obtained

$$\mathscr{K}\boldsymbol{u} = \boldsymbol{f}_i + \boldsymbol{f}_h + \boldsymbol{f}_{eT} \tag{13}$$

The inertial and gyric forces $\boldsymbol{f}_i$ and $\boldsymbol{f}_h$ must include the contributions due to rigid displacement:

$$\boldsymbol{f}_i(\boldsymbol{r}, t) = -\mathscr{M}\ddot{\boldsymbol{w}}, \qquad \boldsymbol{f}_h(\boldsymbol{r}, t) = -\mathscr{G}\dot{\boldsymbol{w}} \tag{14}$$

Hence,

$$\mathscr{M}\ddot{\boldsymbol{w}} + \mathscr{G}\dot{\boldsymbol{w}} + \mathscr{K}\boldsymbol{u} = \boldsymbol{f}_{eT} \tag{15}$$

on E. This motion equation can also be derived using Hamilton's (extended) principle.

Unfortunately, Eq. (15) does not suffice in describing the complete motion of the entire vehicle because, in essence, it considers only the motion of the appendages. This may best be understood by thinking of the terms involving $\boldsymbol{u}_r$ as part of the external force distribution. In other words, Eq. (15) will not yield $\boldsymbol{u}$ independently of $\boldsymbol{u}_r$. To complement Eq. (15), we require motion equations that relate the vehicle as a whole, i.e.,

$$\int_V \ddot{\boldsymbol{w}}\,\mathrm{d}m = \boldsymbol{F}_e$$

$$\int_V \boldsymbol{r}^\times \ddot{\boldsymbol{w}}\,\mathrm{d}m - \tfrac{1}{2}\int_V \boldsymbol{h}_s^\times \nabla^\times \dot{\boldsymbol{w}}\,\mathrm{d}V = \boldsymbol{G}_e \tag{16}$$

where

$$F_e \triangleq \int_V f_{eT}\,dV, \qquad G_e \triangleq \int_V r^{\times} f_{eT}\,dV$$

are the total external force and torque (about O) acting on the vehicle. Equations (15) and (16) complete the dynamical description of the gyroelastic vehicle.

We shall now prepare to use the constrained dynamical parameters to express the general motion of the vehicle. It will be very helpful to work with an equivalent first-order form of Eq. (15). Reshaping Eq. (15), we arrive at the following:

$$\mathscr{M}^1 U_r \ddot{q}_r + \mathscr{G}^1 U_r \dot{q}_r + \mathscr{E}\dot{\chi} + \mathscr{S}\chi = \gamma_e \tag{17}$$

on E, where

$$\mathscr{M}^1 \triangleq \begin{pmatrix} \mathscr{M} \\ \cdot \end{pmatrix}, \qquad \mathscr{G}^1 \triangleq \begin{pmatrix} \mathscr{G} \\ \cdot \end{pmatrix}, \qquad q_r \triangleq \begin{pmatrix} w_0 \\ \theta \end{pmatrix}$$

The definitions for $\mathscr{E}$, $\mathscr{S}$, χ, and γ_e are the same as in Eq. (6).

Let us expand χ as follows:

$$\chi(r,t) = \sum_\beta \Omega_\beta^{-1} \psi_\beta(r) q_\beta(t) \tag{18}$$

For a vehicle with N appendages, the set of constrained eigenfunctions $\{\psi_\alpha\}$ actually comprises N (disjoint) subsets of eigenfunctions $\{\psi_{n\alpha}\}$, one set for each appendage. Thus, it may be more palatable to think of Eq. (18) as

$$\chi(r,t) = \sum_{n=1}^{N} \sum_\beta \Omega_{n\beta}^{-1} \psi_{n\beta}(r) q_{n\beta}(t) \tag{19}$$

We insert Eq. (18) into Eq. (17), premultiply by ψ_α^T, and integrate over E, while observing the orthogonality conditions for ψ_α, to give

$$p_\alpha^T \dot{v}_0 + h_\alpha^T \dot{\omega}_0 - g_\alpha^T \omega_0 + 2\dot{q}_\alpha + 2\Omega_\alpha q_{-\alpha} = \gamma_\alpha \tag{20}$$

where

$$\gamma_\alpha(t) \triangleq \int_E u_\alpha^T f_{eT}\,dV$$

and $v_0 = \dot{w}_0$ and $\omega_0 = \dot{\theta}$. And substituting the "upper half" of Eq. (18) into Eq. (16) yields

$$m\dot{v}_0 - c^{\times}\dot{\omega}_0 + \sum_\alpha p_\alpha \dot{q}_\alpha = F_e \tag{21}$$

$$c^{\times}\dot{v}_0 + J\dot{\omega}_0 + \sum_\alpha h_\alpha \dot{q}_\alpha - h_T^{\times}\omega_0 + \sum_\alpha g_\alpha q_\alpha = G_e \tag{22}$$

where m, c, J are the zeroth (i.e., mass), first, and second moments of inertia (about O) of V.

Introducing the following definitions:

$$P \triangleq \text{row}\{p_\alpha, p_{-\alpha}\}, \qquad H \triangleq \text{row}\{h_\alpha, h_{-\alpha}\},$$

$$G \triangleq \text{row}\{g_\alpha, g_{-\alpha}\}, \qquad q_e \triangleq \text{col}\{q_\alpha, q_{-\alpha}\}, \tag{23}$$

$$\Omega \triangleq \text{diag}\left\{\begin{pmatrix} \cdot & \Omega_\alpha \\ -\Omega_\alpha & \cdot \end{pmatrix}\right\}, \qquad \gamma_e \triangleq \text{col}\{\gamma_\alpha, \gamma_{-\alpha}\}$$

we can economically write Eqs. (20–22) as

$$\begin{pmatrix} m\mathbf{1} & -c^{\times} & P \\ c^{\times} & J & H \\ P^T & H^T & 2\mathbf{1} \end{pmatrix} \begin{pmatrix} \dot{v}_0 \\ \dot{\omega}_0 \\ \dot{q}_e \end{pmatrix} + \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & -h_T^{\times} & G \\ \cdot & -G^T & 2\Omega \end{pmatrix} \begin{pmatrix} v_0 \\ \omega_0 \\ q_e \end{pmatrix} = \begin{pmatrix} F_e \\ G_e \\ \gamma_e \end{pmatrix} \tag{24}$$

The two coefficient matrices are, respectively, symmetric and skew-symmetric. The first matrix is furthermore positive-definite, provided that the vehicle contains a rigid body ($m_r, J_r > 0$). This can be shown using Sylvester's criterion in conjunction with the modal identities.[2] It is worthwhile to compare the derivation of Eq. (24) to that for nongyric elastic vehicles.[7]

The importance and practicality of this formulation should not be overlooked. It is often advantageous to analyze the substructures of a vehicle individually before considering the vehicle *in toto*. This practice is quite common and necessary in the design of complex vehicles, for obvious reasons of efficiency and manageability. Moreover, different substructures often demand different areas of expertise. The above formulation thus offers an avenue of analysis in such cases. The constrained modal parameters can be determined for each substructure independently and then used in consolidating a dynamical model for the unconstrained vehicle.

Let us write Eq. (24) as

$$E\dot{\zeta} + S\zeta = \gamma \tag{25}$$

where $E^T = E > 0$, $S^T = -S$. The corresponding eigenproblem is

$$(\lambda_\alpha E + S)\zeta_\alpha = 0 \tag{26}$$

Owing to the properties of E and S, the eigenvalues are purely imaginary, as should be expected since the system is conservative, and appear in complex conjugate pairs; that is, $\lambda_\alpha = \pm j\omega_\alpha$, where ω_α are the (unconstrained) vibration frequencies. (The indicial system used again assumes $\omega_{-\alpha} = -\omega_\alpha$, $-\infty < \alpha < \infty$, $\alpha \neq 0$.) Although all the vehicle's modal information can be extracted from Eq. (26), a complete discussion of the vibration frequencies and the mode shapes of a gyroelastic vehicle is deferred to the companion paper,[3] which uses another formulation of the problem. The form (26) does, however, provide us with a coign of vantage for investigating *pseudorigid* and *precessional* modes.

Pseudorigid Modes

It will be necessary in what follows to distinguish between rigid *position* modes and rigid *rate* modes. The latter express the ability of the vehicle to move uniformly with respect to certain rigid degrees of freedom, for example, a uniform translation or rotation, given mathematically by

$$w(r,t) \equiv v_r(r)t \tag{27}$$

Position modes, on the other hand, express the ability to position the vehicle arbitrarily with respect to certain rigid degrees of freedom, that is,

$$w(r,t) \equiv u_r(r) \tag{28}$$

The position modes are, in fact, the modes $U_r(r)$ discussed earlier, in Eq. (5). The vibration frequencies corresponding to all these (position and rate) modes are zero.

In the case of a nongyric elastic vehicle, the position-mode shapes and the rate-mode shapes are identical; or, in other words, if a vehicle has a given degree of freedom, then it is able to move uniformly with respect to that degree of freedom. For example, if a (nongyric) vehicle can be oriented in an arbitrary position about an axis $\boldsymbol{a}$, say, then it is free to rotate uniformly about $\boldsymbol{a}$ as well. As we shall see, though, this is generally not so for gyroelastic vehicles; the rate-mode shapes are substantially different from the position-mode shapes. A gyroelastic vehicle, in contrast to a purely elastic one, does not necessarily exhibit zero strain energy, the rigid-body assumption, in its rate modes, even though the associated modal frequencies are zero. In addition to a rigid motion of the vehicle, there is a (constant) elastic deformation. We introduce the adjective *pseudorigid* to describe these modes.

The eigencolumns of the system corresponding to $\lambda_\alpha = 0$, as given by Eq. (26), have the particular form,

$$\zeta_{0\alpha} = \begin{pmatrix} \mathbf{1}_1 \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{1}_2 \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{1}_3 \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{0} \\ \boldsymbol{a} \\ \boldsymbol{q}_{e0} \end{pmatrix} \tag{29}$$

for $h_T \neq 0$. It should be mentioned that the eigenproblem (26) does not yield the position modes since these have already been factored out. This, in fact, is the latent reason why $\boldsymbol{E}$ is positive-definite instead of merely positive-semidefinite. The number of rate modes is equal to the order of the degeneracy of $\boldsymbol{S}$, which is 4 when $h_T \neq 0$. The translational rate-mode shapes $\{\zeta_{01}, \zeta_{02}, \zeta_{03}\}$ are identical to the translation position-mode shapes. This is not surprising since the gyricity distribution can only affect rotational degrees of freedom. The last eigencolumn ζ_{04} in Eq. (29) is a remnant of the three rotational position-mode shapes. If $h_T = 0$, there are three rotational rate modes (since $\boldsymbol{S}$ is degenerate of order 6):

$$\zeta_{0\alpha} = \begin{pmatrix} \mathbf{0} \\ \mathbf{1}_1 \\ \boldsymbol{q}_{e01} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{0} \\ \mathbf{1}_2 \\ \boldsymbol{q}_{e02} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{0} \\ \mathbf{1}_3 \\ \boldsymbol{q}_{e03} \end{pmatrix} \tag{30}$$

Furthermore, if $h_s \equiv 0$, then $\boldsymbol{q}_{e01} = \boldsymbol{q}_{e02} = \boldsymbol{q}_{e03} = \mathbf{0}$.

Insight into the nuances of the rotational mode can be gained by considering the corresponding mode shape. From Eq. (18),

$$\boldsymbol{w}_{04}(\boldsymbol{r}) = -\boldsymbol{r}^{\times}\boldsymbol{a} + \sum_{\alpha} \boldsymbol{u}_\alpha(\boldsymbol{r})\, q_{0\alpha} \tag{31}$$

and, since this is a rate mode, $\boldsymbol{w}_{04}(\boldsymbol{r}, t) = \boldsymbol{v}_{04}(\boldsymbol{r})t$. It can be shown[5] that the sum in Eq. (31) vanishes and, therefore,

$$\boldsymbol{v}_{04}(\boldsymbol{r}) = -\boldsymbol{r}^{\times}\boldsymbol{a} \tag{32}$$

We conclude then that this mode is a uniform rotation of the vehicle about the axis $\boldsymbol{a}$.

The (static) elastic deflections associated with this mode are found from Eq. (18) to be

$$\boldsymbol{u}_{04}(\boldsymbol{r}) = \sum_{\alpha} \Omega_\alpha^{-1} \boldsymbol{v}_\alpha(\boldsymbol{r})\, q_{0\alpha} \tag{33}$$

which is not, in general, zero. Thus, the vehicle is deformed from its nominal state as it rotates about $\boldsymbol{a}$. Moreover, the elastic deflections are proportional to the rate of rotation. These results imply the following conclusion: In general, *there exists for gyroelastic vehicles a mode whose frequency is zero but whose associated (elastic) strain energy is nonzero.*

As an example of the foregoing, consider a vehicle consisting of a rigid body with two identical rods cantilevered to it (Fig. 2a). There is a skew-symmetric gyricity distribution in the rods parallel to the rods and a rotor in the rigid body whose angular momentum $\boldsymbol{h}_0$ is perpendicular to the rods. Clearly, $\boldsymbol{h}_T = \boldsymbol{h}_0$. Thus, the pseudorigid mode for this vehicle is a uniform rotation about $\boldsymbol{h}_0$, with the rods deflected as shown in Fig. 2b.

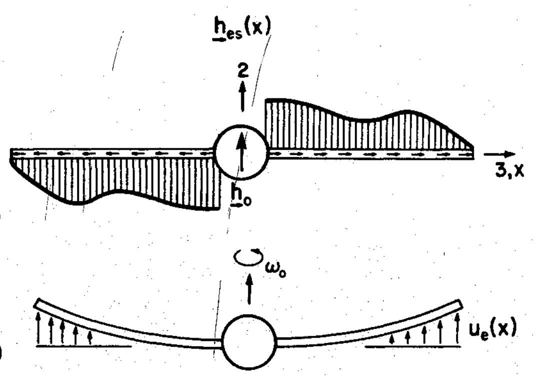

Fig. 2 Pseudorigid mode for a simple gyroelastic spacecraft.

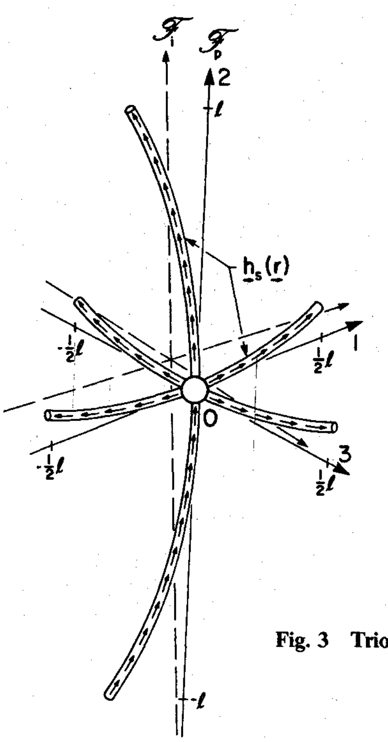

Fig. 3 Triorthogonal-boom spacecraft.

Precessional Mode

At this point, one might wonder what has become of the rate modes of a vehicle which exist in the absence of gyricity. To explain this phenomenon, let us examine the effects on the rate modes, as determined from Eq. (26), due to a small amount of gyricity, that is, $h_s(\boldsymbol{r}) = 0(\epsilon)$.

The eigenequation is given by

$$\det\begin{pmatrix} \lambda \boldsymbol{E}_r + \boldsymbol{S}_r & \lambda \boldsymbol{E}_e + \boldsymbol{S}_e \\ \lambda \boldsymbol{E}_e^T - \boldsymbol{S}_e^T & 2(\lambda \mathbf{1} + \boldsymbol{\Omega}) \end{pmatrix} = 0 \tag{34}$$

or, equivalently,

$$\det(\lambda \mathbf{1} + \boldsymbol{\Omega}) \det\big[\lambda \boldsymbol{E}_r + \boldsymbol{S}_r - \tfrac{1}{2}(\lambda \boldsymbol{E}_e + \boldsymbol{S}_e)(\lambda \mathbf{1} + \boldsymbol{\Omega})^{-1}(\lambda \boldsymbol{E}_e - \boldsymbol{S}_e)^T\big] = 0 \tag{35}$$

where $\lambda = j\omega$. The definitions for $\boldsymbol{E}_r$, $\boldsymbol{E}_e$, $\boldsymbol{S}_r$, and $\boldsymbol{S}_e$ may be inferred from Eq. (24). We shall assume that the eigenvalues that were zero when $h_s(\boldsymbol{r}) \equiv 0$ remain unchanged or become only slightly nonzero, i.e., $\omega = 0(\epsilon)$, when $h_s(\boldsymbol{r})$ becomes slightly nonzero. Hence, to first order, Eq. (35) is

$$\det(j\omega \boldsymbol{E}_r + \boldsymbol{S}_r) = 0 \tag{36}$$

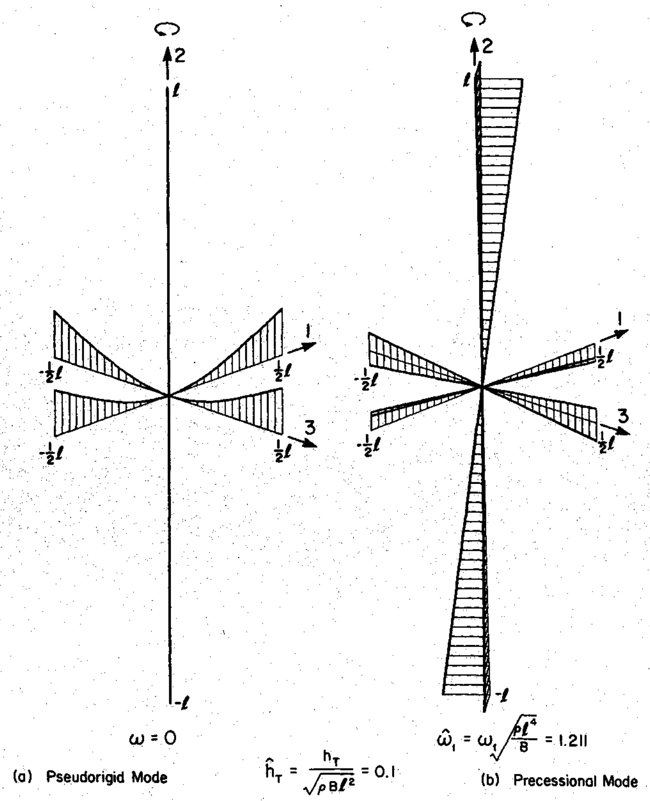

Fig. 4 Pseudorigid and precessional modes for triorthogonal-boom spacecraft.

For $h_T \neq 0$, there is one nonzero frequency pair $\{\pm\omega_p\}$, which is given[8] by

$$\omega_p^2 = \frac{\boldsymbol{h}_T^T \boldsymbol{I} \boldsymbol{h}_T}{\det \boldsymbol{I}} \tag{37}$$

where $\boldsymbol{I}$ is the vehicle inertia matrix in a center-of-mass reference frame and ω_p is known as the *precession frequency*. The corresponding mode will accordingly be called the *precessional mode* (or *mode-pair*) and is similar to the motion of a spinning, precessing rigid body. This accounts for the loss of the rate modes.

Numerical Example

As a numerical example, we consider a triorthogonal-boom model of a spacecraft (Fig. 3). The model consists of one flexible longitudinal boom and two flexible lateral booms, all of uniform and equal density ρ and stiffness B, fastened together at their midpoints. A rigid body is placed at the attachment point. The gyricity distribution, maintained parallel to the booms, is chosen to be

$$h_s(\boldsymbol{r}) = h_T \begin{pmatrix} \frac{\pi}{2\ell}\sin\left(\frac{2\pi}{\ell}r_1\right), & r_2 = r_3 = 0 \\ \frac{1}{\ell}\left(1 - \frac{|r_2|}{\ell}\right), & r_1 = r_3 = 0 \\ \frac{\pi}{2\ell}\sin\left(\frac{2\pi}{\ell}r_3\right), & r_1 = r_2 = 0 \end{pmatrix} \tag{38}$$

where h_T represents the total gyricity. Note that the distribution is skew-symmetric in the lateral booms and, thus, the net stored angular momentum is directed along the longitudinal boom. The pseudorigid and precessional modes were determined using a finite-element analysis[5] and are shown in Fig. 4. (The precessional mode is defined by a pair of mode shapes representing the real and imaginary parts.) As one might expect, the pseudorigid mode is a rotation about the longitudinal boom with the lateral booms deflected. The vehicle in its precessional mode is nearly rigid and precesses about the longitudinal axis.

Concluding Remarks

The use of a continuous distribution of gyricity is a convenient artifice in the analysis of elastic vehicles with a very large number of rotors. The concept is proposed in the same spirit as continuous mass and stiffness distributions are sometimes employed to represent large truss-like lattice structures.

A theory for the dynamics of constrained gyroelastic bodies having been established elsewhere, this paper shows how the general motion of a gyroelastic vehicle can be expressed in terms of the dynamics of its several substructures. Indeed, the resulting dynamical equations may be written in terms of the (constrained) substructural modal parameters obtained by considering the appendages of the vehicle separately. This result is of practical importance since, in reality, the design and analysis of individual appendages often proceed independently of the work done on the rest of the spacecraft.

Momentum wheels and/or control moment gyros distributed over a large flexible spacecraft may well prove to be a powerful and efficient method of shape control. Our analysis has shown that such distributions cause the vehicle to exhibit a pseudorigid mode in which the spacecraft rotates uniformly in a deformed state. This type of mode has not, to our knowledge, been discussed previously in the literature. It may be possible to use these modes effectively in the shape control of antenna reflectors, for example. Under other circumstances, pseudorigid modes may be an undesirable side effect of using rotors as torque actuators. In either case, they must be fully understood; moreover, since pseudorigid modal deformations are proportional to rate of rotation, it may not always be possible to ignore the effects of pseudorigid modes even when a vehicle is only "slightly" gyric.

Acknowledgments

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